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INFLUENCE OF POISSON'S RATIO ON THE CONDITION OF THE FINITE ELEMENT STIFFNESS MATRIX

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Abstract—The dependence of the spectral condition number of the finite element stiffness matrix, in the case of a nearly incompressible solid, is established as a function of the mesh size and Poisson's ratio.

INTRODUCTION

IT HAS been observed [1] in the finite element analysis of elastic solids that as the Poisson ratio ν of the material approaches $\frac{1}{2}$ (or -1), that is as the material nears incompressibility, the global stiffness matrix becomes progressively more ill-conditioned until the matrix becomes computationally singular.

It is the main purpose of this paper to determine the influence of ν, as the material nears incompressibility, on the spectral condition number of the stiffness matrix. This is achieved with a technique originally described in Ref. [2] (see also Refs. [3–6] for bounding the extremal eignevalues of the *global* mass and stiffness matrices in terms of the extremal eigenvalues of the corresponding *element* matrices, the number of elements meeting at a point and the fundamental frequency of the structure.

SPECTRAL CONDITION NUMBER

The method of finite elements reduces the continuous boundary value problem and eigenvalue problem

$$Lu = f \quad \text{and} \quad Lu = \lambda u \quad \text{in } \Omega \tag{1}$$

where L is a linear differential operator and where u satisfies some boundary condition on $\partial\Omega$, to the corresponding algebraic problems

$$KU = b \quad \text{and} \quad KU = \mu MU \tag{2}$$

where K and M are the global stiffness and mass matrices.

The spectral condition numbers $C_2(K)$ of K and $C_2(M)$ of M are defined as

$$C_2(K) = \|K\|_2\|K^{-1}\|_2 \quad \text{and} \quad C_2(M) = \|M\|_2\|M^{-1}\|_2 \tag{3}$$

or since K and M are symmetric and at least positive semi-definite

$$C_2(K) = \lambda_N^K/\lambda_1^K \quad \text{and} \quad C_2(M) = \lambda_N^M/\lambda_1^M \tag{4}$$

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in which λ_1^K, λ_N^K, λ_1^M and λ_N^M are the lowest (1st) and highest (Nth) eigenvalues of K and M.

Bounds will be derived now on the extremal eigenvalues of K and M and consequently on $C_2(K)$ and $C_2(M)$. For this, let k_e and m_e denote the element stiffness and mass matrices. Let also U and u_e denote a global and element vectors; then the quadratics U^TKU and U^TMU can be written as

$$U^TKU = \sum_{e=1}^{N_e} u_e^T k_e u_e \quad \text{and} \quad U^TMU = \sum_{e=1}^{N_e} u_e^T m_e u_e \tag{5}$$

where summation is carried out over all the N_e finite elements in the mesh.

Denoting by $\phi_1, \phi_2, \ldots, \phi_n$ the shape functions inside the element, the quadratic $u_e^T m_e u_e$ can be written as

$$u_e^T m_e u_e = \int_\Delta (u_1\phi_1 + u_2\phi_2 + \ldots + u_n\phi_n)^2 \, dx \, dy \, dz. \tag{6}$$

Hence if the shape functions are *linearily independent* then m_e is *positive definite* such that $\lambda_1^{m_e} > 0$ for all e.

Denoting by λ_1^k, λ_n^k, λ_1^m and λ_n^m the extremal eigenvalues of the element matrices, we have for each element

$$\lambda_1^k u^T u \leq u^T k u \leq \lambda_n^k u^T u \quad \text{and} \quad \lambda_1^m u^T u \leq u^T m u \leq \lambda_n^m u^T u \tag{7}$$

Substituting equation (7) into equation (5) we obtain

$$\min_e(\lambda_1^{k_e}) \sum_{e=1}^{N_e} u_e^T u_e \leq U^TKU \leq \max_e(\lambda_n^{k_e}) \sum_{e=1}^{N_e} u_e^T u_e \tag{8}$$

and the same thing for M.

If U is normalized such that $U^TU = 1$ then it can be shown that

$$1 \leq \sum_{e=1}^{N_e} u_e^T u_e \leq p_{\max} \tag{9}$$

where $p_{\max}$ denotes the maximum number of elements meeting at a nodal point.

Equations (8) and (9) lead to

$$\begin{aligned} \max_e(\lambda_n^{k_e}) &\leq \lambda_N^K \leq p_{\max} \max_e(\lambda_n^{k_e}) \\ \max_e(\lambda_n^{m_e}) &\leq \lambda_N^M \leq p_{\max} \max_e(\lambda_n^{m_e}) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \min_e(\lambda_1^{k_e}) &\leq \lambda_1^K \leq \lambda_N^K \\ \min_e(\lambda_1^{m_e}) &\leq \lambda_1^M \leq \lambda_N^M. \end{aligned} \tag{11}$$

But since the element stiffness matrix k is usually only positive semi definite, the lower bound on λ_1^K as given in equation (11) is reduced to the trivial fact that $\lambda_1^K \geq 0$.

To obtain a non trivial bound on λ_1^K we make use of the variational nature of the finite element method and Rayleigh's principle. This principle asserts that if λ_1 is the lowest exact eigenvalue of the structure then

$$U^T K U / U^T M U \geq \lambda_1. \tag{12}$$

Choosing U to correspond to λ_1^K we obtain from equation (12) that

$$\lambda_1^K \geq \lambda_1 \lambda_1^M \tag{13}$$

and consequently from equation (11) that

$$\lambda_1^K \geq \lambda_1 \min_e(\lambda_1^{m_e}). \tag{14}$$

Also [2]

$$\lambda_1^K \leq \mu_1 p_{\max} \max_e(\lambda_n^{m_e})$$

where μ_1 is the lowest eigenvalue calculated by the finite element method. For a sufficiently fine mesh μ_1 will be close [7] enough to λ_1 and replacing μ_1 by λ_1 we have

$$\lambda_1 \min_e(\lambda_1^{m_e}) \leq \lambda_1^K \leq \lambda_1 p_{\max} \max_e(\lambda_n^{m_e}). \tag{15}$$

The bounds on $C_2(K)$ and $C_2(M)$ become then

$$\frac{\max(\lambda_n^k)}{\lambda_1 \max(\lambda_n^m) p_{\max}} \leq C_2(K) \leq \frac{\max(\lambda_n^k) p_{\max}}{\lambda_1 \min(\lambda_1^m)} \tag{16}$$

and

$$1 \leq C_2(M) \leq \frac{\max(\lambda_n^m) p_{\max}}{\min(\lambda_1^m)}. \tag{17}$$

Equations (16) and (17) are the principal results of this section.

Since λ_1 appearing in equation (16) is the fundamental eigenvalue of the continuous structure and is therefore independent of the discretization, the dependence of the bounds on $C_2(K)$ and $C_2(M)$ on the discretization parameters is expressed solely by λ_1^m, λ_n^m, λ_n^k and $p_{\max}$. It is also seen from equations (16) and (17) that a sufficient condition for the invertibility of K and M is the positive definiteness of m. This is assured in turn by the linear independence of the shape functions.

As an application to equations (18) and (19) consider a triangular membrane element with linear variation of displacements inside it. Its element stiffness and mass matrices can be written as

$$k_{ij} = \frac{1}{2A} h_i h_j n_i^T n_j \qquad i, j = 1, 2, 3, \qquad \text{and } m = \frac{A}{6}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{18}$$

where A, h_i and n_i are the area of the element its sides and unit vectors normal to the sides, respectively. Here

$$1/\sin\theta \leq \lambda_N^K \leq 3p_{\max}/\sin\theta \tag{19}$$

$$\lambda_1 A_{\min}/6 \leq \lambda_1^K \leq 2\lambda_1 p_{\max} A_{\max}/3 \tag{20}$$

in which θ denotes the smallest angle in the mesh. Hence

$$\frac{3}{2\lambda_1 p_{\max} A_{\max} \sin\theta} \leq C_2(K) \leq \frac{18 p_{\max}}{\lambda_1 A_{\min} \sin\theta} \tag{21}$$

and the condition of the stiffness matrix inevitably deteriorates as θ is decreased.

In the same manner we obtain for the tetrahedron

$$\frac{H_{\max}}{3\lambda_1 p_{\max} V_{\max}} \leq C_2(K) \leq \frac{30 p_{\max} H_{\max}}{\lambda_1 V_{\min}} \tag{22}$$

where V denotes the volume of the element and where H = area of largest face/volume.

MAXIMUM CONDITION NUMBER

The maximum condition number $C_\infty(K)$ is defined as

$$C_\infty(K) = \|K\|_\infty \|K^{-1}\|_\infty \tag{23}$$

where

$$\|K\|_\infty = \max_i \sum_{j=1}^{N} |K_{ij}|. \tag{24}$$

Obtaining a bound on $\|K\|_\infty$ is simple and we readily have

$$\|K\|_\infty \leq p_{\max} \max_e \|k_e\|_\infty. \tag{25}$$

For a positive definite matrix K of dimension N, $K_{ii}+K_{ij} > |K_{ij}|$ for any i and j. Therefore

$$\|K\|_\infty < N \max_i(K_{ii}). \tag{26}$$

The finite element solution $\hat{u}$ is obtained by minimizing the total potential energy $\pi(u)$

$$\pi(u) = a(u, u) - (f, u) \tag{27}$$

where $a(u, u)$ is the (say elastic) energy and where (f, u) is the work of the external forces f.

Also

$$\pi(\hat{u}) - \pi(u) = a(u-\hat{u}, u-\hat{u}) \geq 0 \tag{28}$$

and since the first variation of π at u and $\hat{u}$ vanishes we have

$$(f, u-\hat{u}) \geq 0 \tag{29}$$

or

$$(f, u) \geq (f, \hat{u}). \tag{30}$$

Choosing f to be a point force (delta function) equation (30) yields $u \geq \hat{u}$. That is, the true solution u at a point of application of a point force is always larger or equal to the finite element displacement $\hat{u}$ at that point. The response (Green's) function to a point force at ξ is denoted by $G(x, \xi)$. We also denote by Γ the maximum of $G(x, x)$. The diagonal terms K_{ii}^{-1} of the flexibility matrix are the responses to point loads and therefore

$$\max_i(K_{ii}^{-1}) \le \Gamma \qquad i = 1, 2, \ldots, N. \tag{31}$$

Hence from equation (26) we have

$$\|K^{-1}\|_\infty < N\Gamma \tag{32}$$

and consequently

$$C_\infty(K) < N\Gamma p_{\max} \max_e \|k_e\|_\infty. \tag{33}$$

The maximum of the influence function Γ in equation (33) plays the role of λ_1 in equation (16).

More on this can be found in Ref. [6]. Since Γ is in many cases unbounded the bound in equation (33) is less general than that in equation (16).

INFLUENCE OF POISSON'S RATIO

The bounds on $C_2(K)$ in equation (16) become closer as λ_n^m/λ_1^m nears 1. These bounds are therefore particularly suitable to study the influence on $C_2(K)$ of such discretization and intrinsic parameters of the problem that do not enter into the ratio λ_n^m/λ_1^m. This is the case with Poisson's ratio v which being an elastic property of the material appears only in the element stiffness matrix k but not in the element mass matrix m. We should be able then, by using equation (16), to obtain sharp bounds on $C_2(K)$ and hence to establish the influence of v on the condition of K.

We consider first a three dimensional solid discretized by a uniform mesh of rectangular tetrahedronal elements.

The three displacements u, v and w are interpolated inside the element by

$$\begin{aligned} u &= u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + u_4\phi_4 \\ v &= v_1\phi_1 + v_2\phi_2 + v_3\phi_3 + v_4\phi_4 \\ w &= w_1\phi_1 + w_2\phi_2 + w_3\phi_3 + w_4\phi_4 \end{aligned} \tag{34}$$

where u_i, v_i and w_i, $i = 1, 2, 3, 4$ are the nodal values of u, v and w and where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are given by

$$\phi_1 = 1-(x+y+z)/h,\ \phi_2 = x/h,\ \phi_3 = y/h \text{ and } \phi_4 = z/h. \tag{35}$$

The element stiffness matrix k are computed from the elastic energy expression

$$\frac{1}{2}\int_\Omega (\Lambda e^2 + 2G(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + G(e_{xy}^2 + e_{yz}^2 + e_{xz}^2))\,dx\,dy\,dz$$

in which e_{xx}, e_{yy} and e_{zz} are the direct strains, e_{xy}, e_{yz}, e_{xz} are the shear strains and e is the volume expansion. The relation between Poissons ratio v, the elastic modulus E and Λ and G is given by

$$\Lambda = \frac{Ev}{(1+v)(1-2v)} \quad \text{and} \quad G = \frac{E}{2(1+v)}. \tag{36}$$

as $\nu \uparrow \frac{1}{2}$ the mode corresponding to the maximal eigenvalue of the element stiffness matrix becomes nearly that producing a pure volume change. Now, since

$$e = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \tag{37}$$

the dilatation mode is $(-1, 1, 0, 0, -1, 0, 1, 0, -1, 0, 0, 1)$ and the maximum eigenvalue of k, for $\nu \uparrow \frac{1}{2}$ is given by

$$\lambda_{12}^k = 6\Lambda V/h^2. \tag{38}$$

Hence

$$\lambda_{12}^k = \frac{E\nu h}{(1+\nu)(1-2\nu)}. \tag{39}$$

For $\nu \downarrow -1$ we obtain in the same manner

$$\lambda_{12}^k = \frac{6Eh}{(1+\nu)(1-2\nu)}. \tag{40}$$

The element mass matrix m for the tetrahedronal element can be written as

$$m = \frac{h^3}{60} \begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix}, \quad a = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \tag{41}$$

Its extremal eigenvalues being $\lambda_1^m = h^3/60$ and $\lambda_{12}^m = h^3/12$. Hence

$$c_1 \frac{E}{\lambda_1 p_{\max}} \frac{h^{-2}}{(1+\nu)(1-2\nu)} \le C_2(K) \le c_2 \frac{E p_{\max}}{\lambda_1} \frac{h^{-2}}{(1+\nu)(1-2\nu)} \tag{42}$$

in which $c_1 = 6$ and $c_2 = 30$ for $\nu \uparrow \frac{1}{2}$, and $c_1 = 72$ and $c_2 = 360$ for $\nu \downarrow -1$. The lowest eigenvalue λ_1 of the elastic solid is proportional to E and may well depend on ν. It is certain, however, that it remains bounded as the solid nears incompressibility. As c_1, c_2 and λ_1 are finite at the limit $\nu \uparrow \frac{1}{2}$ or $\nu \downarrow -1$, equation (46) predicts that

$$C_2(K) = c \frac{h^{-2}}{(1+\nu)(1-2\nu)} \tag{43}$$

where c is independent of h and ν.

Generalizing these results to higher order elements is formal. The minimal eigenvalue of m is generally given by $\lambda_1^m = c_3 h^3$ where c_3 is independent of h and ν. The maximum eigenvalue of k is given by $\lambda_n^k = c_4 Eh/(1+\nu)(1-2\nu)$ where c_4 is again independent of h and ν. Therefore also in the more general case the bounds in equation (42) and consequently equation (43) hold.

CONCLUSIONS

Let λ_1^k, λ_n^k, λ_1^m and λ_n^m be the extremal (1st and nth) eigenvalues of the *element* stiffness and mass matrices k and m. Let also λ_1 be the *exact* lowest eigenvalue of the structure and

p_{max} the maximum number of finite elements meeting at a nodal point. Then the spectral condition number $C_2(K)$ of the stiffness matrix K and $C_2(M)$ of the mass matrix M are bounded by

$$\frac{\max(\lambda_n^k)}{\lambda_1 \max(\lambda_n^m)p_{max}} \leq C_2(K) \leq \frac{\max(\lambda_n^k)p_{max}}{\lambda_1 \min(\lambda_1^m)} \tag{44}$$

and

$$1 \leq C_2(M) \leq \frac{\max(\lambda_n^m)p_{max}}{\min(\lambda_1^m)} \tag{45}$$

where max( ) and min( ) refer to maximal and minimal values in the mesh.

Let Γ denote the exact maximum deflection due to a point load (a point torque etc.) at the point of application. Then

$$C_\infty(K) < N\Gamma p_{max} \max(\|k\|_\infty). \tag{46}$$

For a nearly incompressible solid the bounds in equation (44) yield

$$c_1 \frac{Eh^{-2}}{\lambda_1 p_{max}(1+\nu)(1-2\nu)} \leq C_2(K) \leq c_2 \frac{Ep_{max}h^{-2}}{\lambda_1(1+\nu)(1-2\nu)} \tag{47}$$

where ν is Poisson's ratio, h the diameter of the element, and c_1 and c_2 are independent of h and ν. For a right angular tetrahedronal element, and $\nu \uparrow \frac{1}{2}$, $c_1 = 6$ and $c_2 = 30$. A typical value for E/λ_1 (for a nearly spherical [8] solid of diameter 1) is 1.

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Абстракт—Для случая близи иесжимаемого твердого тела, определяется зависимость числа спектрального условия для матрицы коэффициентов жесткости в методе конечного элемента, в виде функции размера отверстия и числа Пуассона.